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by

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Summary

Miner's rule for the cumulative damage due to fatigue, a deterministic formula which is well known in engineering practice, has been examined earlier from a probabilistic point of view with Birnbaum in [2]. Here the assumptions of that model are weakened. Previously the basic assumptions were that crack growth was stochastic in nature with incremental extensions having a distribution with increasing failure rate, and that the cycle of load fluctuations was fixed and then repeated under program. We now assume, instead of the distributions of incremental crack extension having increasing failure rate, only that for a given load fluctuation, the expected residual damage increment either in crack initiation or extension, given the damage exceeds a preassigned amount is less than the damage increment which was expected for that load fluctuation before it was imposed. We also weaken the assumptions concerning the type of loading spectra which are admitted, considering the case of random load fluctuations which are cyclic in distribution.

Utilizing results from renewal theory we study the expected number of cycles until failure under both programmed and random loading spectra and exhibit conditions of dependence upon load history under which a generalization of Miner's rule agrees with the mathematical expectation of fatigue life. Under other conditions of dependence we obtain bounds for the expected number of cycles to failure under both programmed and random loading spectra.

1. Introduction

In an earlier publication [2], Birnbaum and Saunders gave a statistical interpretation of Miner's rule (a deterministic formula appearing in [6]) which has been used historically under almost all conditions in fatigue analysis. In that paper Miner's rule was shown, under certain plausible assumptions, to be the mathematical expectation of a stochastic variable regarded as fatigue life. This stochastic variable was the number of periods of cyclic oscillations necessary to force the fatigue crack, of which the incremental extensions per cycle themselves were regarded as random variables of a given type, to exceed a critical crack length. This critical length was also subject to chance fluctuations due to various causes.

In what follows we will consider only standardized specimens of a material which are subjected to fluctuating stresses due to loading of a periodic or stochastic nature. To be more specific, for a load (or load function) we mean a continuous piecewise linear function on the positive real axis, the value of which at any time gives the stress imposed by the deflection of the material specimen. This loading function will be generically denoted by λ, with or without affixes. Thus a load function determines such parameters as maximum stress, minimum stress and average stress which are usually used to define each loading oscillation.

A detailed comparison has been made in [2] among several sets of assumptions and their consequences for Miner's rule in its traditional

form. We quote here only the most frequently used deterministic form of Miner's rule, namely that fatigue life under a spectrum of loads can be expressed as a harmonic mean of the lives under the repetition of certain fixed loads which comprise the spectrum. This form is dependent upon the assumption that the order of the load oscillations can be permuted in any cycle with the same resulting fatigue damage. Thus it becomes necessary to count only the number of oscillations of each load of a given kind.

If a given cycle contains various numbers of different oscillations, say n_i oscillations of the i^{th} load type among k distinct types, then the number of such cycles which can be repeated until failure is

$$N = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{N_i}}, \tag{1.1}$$

where N_i is the number of oscillations to failure under repeated application of the same i^{th} load.

By this cumulative damage rule we determine the fraction of damage accrued during one cycle and use its reciprocal to calculate the total life. In practice N_i are determined from available data on the regression of stress versus number of cycles to failure. The n_i are calculated from a typical spectrum of loads during the cycle. Then Miner's rule in the form given in (1.1) is used to determine the life N in cycles to failure.

The objection that has been made to Miner's rule is that under certain

programmed loads within laboratory control, the rule can predict life either conservatively or unconservatively depending in part upon the order in which the loads are applied. This is to say that the average life can be made significantly less or greater than that predicted by (1.1) by selecting and repeating certain sequences of loads of increasing or decreasing magnitudes. The evidence for this type of behavior is well known, see [7] and the references given there.

What we do subsequently in this paper is to derive a formula for the expected life which retains the influence of load order. The basic assumption made previously concerning the class of random variables governing incremental damage is relaxed here. At the same time the stochastic nature of the imposed loads is taken into account. Moreover, our formula also reduces to the classical form of Miner's rule (1.1) when the loads are deterministic and their order can be neglected.

2. A Probabilistic Model

As in [2], it is assumed that fatigue failure is due to the initiation, growth and ultimate extension of a dominant crack. At each oscillation of the imposed stress, this crack is either being formed by the piling up of atomic dislocations or being extended by the rupture of molecular bonds. In any case the damage accumulates by an amount which is a random function of the magnitude of the imposed stress and the geometry of the specimen, as well as the inhomogeneity of the material and the influence of environment. The incremental damage at each fluctuation is therefore a non-negative random variable

whose distribution may depend upon several unknown parameters, the nature of which we do not specify at present. In what follows we shall speak only of "crack extension" but it is to be kept in mind that we refer not only to the crack growth phase in the usual sense but the sub-microscopic phase of crack initiation as well.

Let λ^i denote a *load history* through i load fluctuations. That is λ^i denotes that portion of the load function λ which extends from time zero when the load was relaxed and the stress zero until the time when the i^{th} fluctuation has occurred.

Our first assumption is:

1° The i^{th} incremental crack extension during the last fluctuation of the loading history λ^i is a non-negative random variable $Z^i(\lambda^i)$, depending only upon λ^i. The $Z^i(\lambda^i)$ for $i \geq 1$ are statistically independent random variables.

This assumption implies the statistical independence of the crack extensions in each fluctuation not only from each other but from the total crack length as well. Of course we do not preclude functional dependence between the successive distributions. In this manner the dependence upon the order of the loads is retained. This assumption appears to be sufficient in most cases. Certainly such dependence is realistic in the initial stages of fatigue crack growth, and may be even in situations such as those in which the stress is near the ultimate yield stress of the metal or the crack long relative to the specimen under test.

We now introduce the nomenclature: If X is a non-negative random variable, its complementary distribution function or *co-distribution* R is defined as unity minus its distribution or

$$R(t) = P[X > t] \quad \text{for} \quad t > 0.$$

In life studies the co-distribution is called either the reliability or the survival distribution. We have chosen another name because such terms would be meaningless in our application here.

We also make an assumption about the probabilistic behavior of the incremental crack extension during a given load fluctuation, considering that the crack extension is influenced by the loading history of preceding fluctuations. Specifically we assume

2° The incremental growth random variable $Z(\lambda)$, for any loading history λ, has a co-distribution $R(\cdot:\lambda)$ which satisfies the inequality, for all $x > 0$

$$R(x:\lambda) \int_0^\infty R(t:\lambda)dt \geq \int_0^\infty R(t+x:\lambda)dt. \tag{2.1}$$

The inequality (2.1) is the definition of a class of distributions described as "new better than used in expectation" and denoted by the acronymn NBUE. It can be interpreted as requiring the expected residual growth of the crack per fluctuation, knowing the crack extension exceeds x, is less than or equal the expected crack growth per fluctuation for any $x > 0$. We can write this, for all $x > 0$, as,

(2.2) $$E[Z(\cdot)-x \mid Z(\cdot) > x] \leq EZ(\cdot)$$

which is equivalent with (2.1).

This concept of NBUE was first introduced by Barlow and Proschan in reliability studies in [1] and was named and discussed systematically by Marshall and Proschan in [4].

We believe that the NBUE assumption is realistic since we can now make one compelling argument which covers both the initiation and crack extension phases of fatigue. We have only to interpret (2.2) in words and note its reasonableness; if a given amount of damage is known to have occurred as a result of a particular stress fluctuation then any expected amount of damage remaining would be less than the total amount of damage expected were that same fluctuation reimposed under identical conditions.

In the previous study, reported in [2], an argument is made that crack growth by the successive rupture of molecular bonds should be a random variable with increasing failure rate (IFR). As we now show, it would then *a fortiori* be NBUE.

We can see easily that IFR class contains the NBUE class if we consider the intermediate classification "new better than used" (NBU) random variables. In this case we must have for each load history

(2.3) $$R(t:\cdot)R(x:\cdot) \geq R(t+x:\cdot) \qquad \text{for all} \quad t,x \geq 0.$$

By integrating both sides of (2.3) with respect to t we obtain the NBUE condition of (2.1). The inequality (2.3) means that the random

residual crack growth knowing the crack has already extended a length $x > 0$ during a fluctuation is stochastically smaller than the random crack growth from the beginning of the fluctuation. The relationship (2.3) above can be rewritten as

$$\frac{R(x:\lambda)-R(t+x:\lambda)}{R(x:\lambda)} \geq 1 - R(t:\lambda). \tag{2.4}$$

The left-hand side of (2.4) being an increasing function of $x > 0$ for each $t > 0$ is equivalent with the general definition of IFR random variables, namely, that $-\ln R(t:\lambda)$ is convex for $t > 0$.

Remark: It follows from 1° that the total crack length at the end of the n^{th} cycle, say S_n, is a random variable which is the sum of independent random variables. Each summand of S_n represents the crack growth during the corresponding cycle.

We make another assumption which replaces the classical one, at present considered to be erroneous, that the load oscillations may be permuted in any cycle without altering the resulting fatigue damage, see [7].

This assumption allows the resolution of a load history into an equivalent one in terms of fatigue data.

3° There exists a finite set of loading oscillations, say $\Omega = \{\omega_0,\ldots,\omega_k\}$, such that for any admissible loading history λ there exists an equivalent $\omega_j \in \Omega$, written $\lambda \sim \omega_j$, for which in distribution

$$Z(\lambda) = Z(\omega_j). \tag{2.5}$$

In particular ω_0 is the loading oscillation such that $Z(\omega_0) = 0$.

When (2.5) holds, the incremental stochastic crack growth resulting from the last oscillation of the load history λ is the same in distribution as the incremental crack growth following the single oscillation ω_j preceeded by any number of repetitions of itself. Thus Ω determines a partition of the set of admissible loading histories into equivalence classes. Since Ω is finite, there exists a fixed number of previous oscillations beyond which the "memory" of Z does not extend. Also note that if $\lambda_1 = \lambda_2$, i.e. both equivalent to the same element in Ω, it does not follow that the two histories are of the same length.

The determination of Ω is not a mathematical problem. Its determination can be regarded as the subject of much of the recent research on the influence of load order on crack growth. In particular work given in [3], [5] and [8] would be of that nature.

We now make an assumption concerning the critical crack length, the occurrence of which defines failure. This critical length can, in practice, mean anything from "catastrophic rupture occurring" to "the crack becoming of such a length as to be visually inspectable". For this reason we take W, the critical crack length, to be a random variable such that

4° The critical crack size W is statistically independent of the crack length S_n at the end of the n^{th} cycle for each $n=1,2,\ldots$.

This means simply that knowing the length W at which the crack will become critical, i.e. failure will take place, gives no information about the stochastic behavior of crack growth and vice versa.

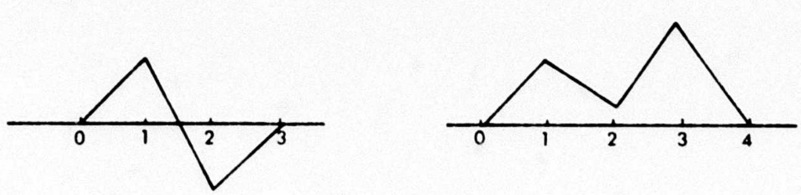
3. Programmed Loads

Let $\mathcal{L}$ be a subset of the continuous real valued piecewise linear functions on the real line each taking the value zero except for some interval of the form (0,m) where m is an integer. We call each element $\lambda \in \mathcal{L}$ a *load spectrum* whenever the salient points of λ can occur only at the integers 1,2,3,... and (primes denoting derivative)

$$\operatorname{sgn} \lambda'(x) = -\operatorname{sgn} \lambda'(x+1) \quad \text{for all non-integral } x > 0.$$

The least positive integer m such that $\lambda(x) = 0$ for $x \geq m$ is called the *length* of the spectrum. Each unit interval across which the load function is not zero we call a load *fluctuation*. The length is merely the number of load fluctuations. Two successive fluctuations, on one of which the load is increasing, is an *oscillation*. We make the arbitrary convention that the number of positive fluctuations is the number of oscillations.

We give an example in figure 1 of two spectra each of two oscillations.



Note that the load oscillations are taken here to be of the same duration, i.e. the frequency is the same. In actual fact with most imposed loads they may not be, but because of the lack of frequency effect in fatigue cycling we shall conveniently make this assumption. This

discussion and justification has been made before by Schijve [8].

In what follows we will require the definition of *continuation* of load function. Consider $\lambda_1, \lambda_2 \in \mathcal{L}$ with m_1 the length of λ_1. We define the binary operation $\lambda_1 * \lambda_2$ on $\mathcal{L}$ by:

$$\lambda_1 * \lambda_2(x) = \lambda_1(x) + \lambda_2(x-m_1)$$

if $\lambda_1(m_1-1)$ and $\lambda_2(1)$ are of same sign and

$$\lambda_1 * \lambda_2(x) = \lambda_1(x) + \lambda_2(x-m_1+1)$$

if $\lambda_1(m_1-1)$ and $\lambda_2(1)$ are of different sign.

We define a programmed load as the repetition of a spectrum. It is thus a cyclic continuous real valued piecewise linear function on the real line of infinite length.

Let $\lambda \in \mathcal{L}$ be given with length $m > 0$. Then $\underline{\lambda}$ is the *programmed load* with spectrum λ whenever $\underline{\lambda} = \lambda_1 * \lambda_2 * \ldots$ where $\lambda_1 = \lambda$ and $\lambda_{j+1}(x) = \lambda_j(x-m)$ for $i = 1, 2, \ldots,$.

Thus λ_j represents the j^{th} cycle of the programmed spectrum. Let λ^i_{j+1} be the partial repetition up through the i^{th} load fluctuation of the $(j+1)^{st}$ cycle, that is λ^i_{j+1} is the restriction of λ_{j+1} in domain to the interval $(jm, jm+i)$. Hence λ^i_{j+1} is the load history of the $(j+1)^{st}$ cycle up through the i^{th} load fluctuation for each $i \geq 1$.

Following the same general approach used previously in [2] we wish to establish bounds for the expected number of cyclic repetitions of $\lambda \in \mathcal{L}$ which can be performed until failure, as determined by various assumptions concerning the stochastic nature of fatigue. We would then

like to compare these with the number given by Miner's Rule.

We make use of 1° to write the random crack extension under the j^{th} cyclic repetition of the load spectrum λ, of length m, as

(3.1) $$Y_j(\lambda) = \sum_{i=1}^{m} Z_j^i(\lambda^i) \qquad j=1,2,\ldots$$

where different affixes i and j on the Z_j^i's indicate independent replications of the corresponding random variables.

Let $N_{Y(\lambda)}$ be the random number of such cycles until failure. It is defined by

(3.2) $$[N_{Y(\lambda)} = n] = [S_{n-1}(\lambda) < W, \; S_n(\lambda) \geq W]$$

where

(3.3) $$S_n(\lambda) = \sum_{j=1}^{n} Y_j(\lambda) \qquad \text{for} \quad n \geq 1.$$

We now state a fundamental

Lemma 1: If $Y_1, Y_2, \ldots$ are independent and identically distributed non-negative random variables with mean μ and $S_n = Y_1 + \cdots + Y_n$ is independent of W for all $n=1,2,\ldots$ then the integer valued random variable $N_Y(W)$ defined by

(3.4) $$[N_Y(W) = n] = [S_{n-1} < W, \; S_n \geq W]$$

always satisfies the left-hand inequality below

(3.5) $$\frac{EW}{\mu} - 1 \leq EN_Y(W) \leq \frac{EW}{\mu}$$

and if the Y_j are NBUE it satisfies the right-hand inequality as well.

Proof: If we let $W = w$ with probability one then the inequality (3.5) is well known to be true under the conditions stated, see [1]. From the independence of W and S_n, (3.5) follows by conditional expectation. A more complete exposition of this point was made in [2].||

By Assumption 1° the $Z_j^i(\lambda^i)$ for all $i = 1,\ldots,m$, are independent and by 2° are NBUE. Thus by a known result, see [4] or as can be easily proved from the definition (2.2), the convolution of such random variables is NBUE. Hence the $Y_j(\lambda)$ for $j \geq 1$ are all random variables which are NBUE. Moreover, they are independent and identically distributed. Thus from Lemma 1 we have, setting

$$w = EW, \quad \nu(\lambda) = EN_{Y(\lambda)} \quad \text{and} \quad EY_j(\lambda) = \mu(\lambda)$$

the relation

$$\frac{w}{\mu(\lambda)} - 1 \leq \nu(\lambda) \leq \frac{w}{\mu(\lambda)} . \tag{3.6}$$

From (3.6) there follows

$$\frac{w}{\nu(\lambda)+1} \leq \mu(\lambda) \leq \frac{w}{\nu(\lambda)} .$$

If a spectrum λ has m fluctuations then λ^i for $i=1,\ldots,m$ denotes the m load histories, which may not all be distinct in their effect upon crack growth. In fact, it is well accepted that the crack does not grow when the load fluctuation is decreasing, see [5]. Additionally, we make the notational convention that

$$\lambda^i \sim \omega_0 \quad \text{for} \quad i > m. \tag{3.7}$$

We may regard each $\omega_j \in \Omega$ for $j=1,\ldots,k$ as a spectrum consisting of a single oscillation per cycle. In this case $Z(\omega_j) = Y(\omega_j)$ and we set

$$(3.8) \qquad EY(\omega_j) = \mu_j \qquad \nu_j = EN_{Y(\omega_j)} \qquad j=1,\ldots,k.$$

This information is usually obtained from the regression plot of cycles to failure versus points in Ω. This is the so-called S-N diagram.

Then by applying the fundamental lemma to the spectrum ω_j we have

$$(3.9) \qquad \frac{w}{\nu_j+1} \leq \mu_j \leq \frac{w}{\nu_i} \qquad j=1,\ldots,k.$$

We now define the number of λ^i in λ equivalent with ω_j. Let $\{\cdot\}$ denote the indicator function taking the value one if true and zero otherwise. Then

$$(3.10) \qquad n_j(\lambda) = \sum_{i\geq 1} \{\lambda^i \simeq \omega_j\} \quad \text{and} \quad z^i(\lambda^i) = \sum_{j=0}^{k} z^i(\omega_j)\{\lambda^i \simeq \omega_j\} .$$

By taking expectations of (3.1) and using (3.10) we see

$$(3.11) \qquad \mu(\lambda) = \sum_{j=1}^{k} \mu_j n_j(\lambda)$$

and substituting from (3.9) obtain

$$(3.12) \qquad w \sum_{i=1}^{k} \frac{n_j(\lambda)}{\nu_j+1} \leq \mu(\lambda) \leq w \sum_{j=1}^{k} \frac{n_j(\lambda)}{\nu_j} .$$

By rearranging we obtain bounds on $w/\mu(\lambda)$ which when substituted into (3.6) yield the following:

Theorem 1: For Programmed Spectra. If the stochastic nature of incremental crack growth satisfies Assumptions 1° and 2°, then each cyclic repetition of a programmed spectrum λ, results in random crack extensions which are independent replications of $Y(\lambda)$. It follows the number of times λ can be repeated until the crack exceeds a given stochastic limit, satisfying 4°, has finite expectation $EN_{Y(\lambda)}$. By Assumption 3° there exists a sufficiently inclusive set Ω of load oscillations for which the information in (3.8) and (3.10) is provided from an S-N diagram for each $\omega_j \varepsilon \Omega$.

Bounds for the expectation are then given by

$$\frac{1}{\sum_{j=1}^{k} \frac{n_j(\lambda)}{\nu_j}} - 1 \le EN_{Y(\lambda)} \le \frac{1}{\sum_{j=1}^{k} \frac{n_j(\lambda)}{\nu_j+1}} \,. \tag{3.13}$$

Note that these are the same bounds given previously in [2] except under weaker conditions on the stochastic nature of crack growth and the broadening of the interpretation of $n_j(\lambda)$.

4. Convex Spaces of Random Spectra

Having weakened the basic Assumptions 1° and 2° by replacing the IFR class of distributions by the NBUE class and given a broader interpretation to the weighting factors $n_j(\lambda)$ appearing in Miner's Rule, we now wish to relax the assumption that the loads which are to be imposed are cyclic in the deterministic sense used hithertofore. Consequently, we make an assumption more in keeping with the stochastic nature of fatigue cycles as they appear in practical situations. By this we mean the cycle

only roughly repeats itself while the individual fluctuations of load do not. We utilize subsequently the convention that random variables will be denoted by upper case letters and the corresponding lower case letters the observed values.

Let us set

$$(4.1) \qquad \mathcal{L}^+ = \{\lambda \in \mathcal{L}: \lambda(x) \geq 0 \text{ for all } x \geq 0\}.$$

If $\lambda \in \mathcal{L}^+$ is not identically zero then it must have even length $m \geq 2$ so without loss of generality assume the first fluctuation is on $(0,1)$, λ is increasing thereon and moreover is then increasing on the intervals $(2j,2j+1)$ for $j=0,\ldots,\frac{m}{2}-1$ and decreasing on the intervals $(2j-1,2j)$ for $j=1,\ldots,\frac{m}{2}$.

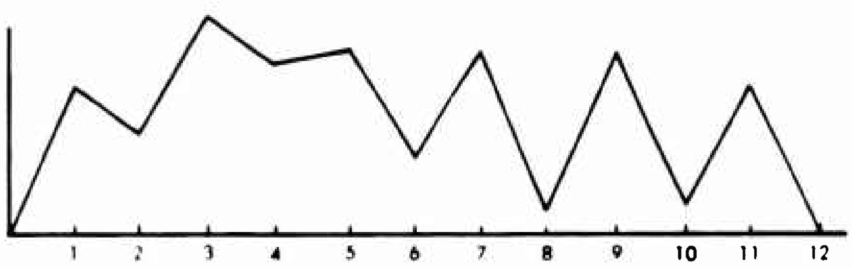
As a consequence of (4.1), λ can be characterized by the vector

$$(4.2) \qquad (\lambda(1),\lambda(2),\ldots,\lambda(m-1))$$

where

$$0 \leq \lambda(2j) < \lambda(2j+1) > \lambda(2j+2) \qquad j=0,\ldots,\frac{m}{2}-1.$$

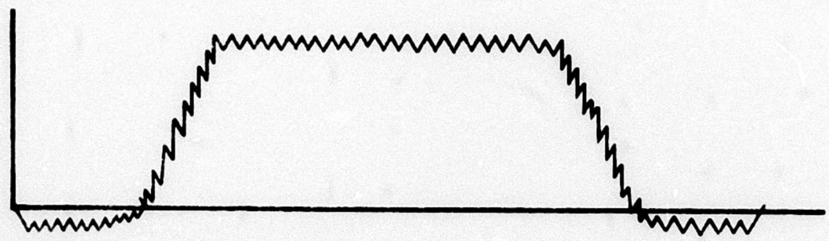
A typical element in $\mathcal{L}^+$ is graphed below in figure 2.



If we denote $\mathcal{L}^- = \{\lambda \in \mathcal{L}: -\lambda \in \mathcal{L}^+\}$, then a space which represents the ground-air-ground cycle in certain aeronautical fatigue studies is a subset of

$$\mathcal{L}^- * \mathcal{L}^+ * \mathcal{L}^-. \tag{4.3}$$

A typical element of this space is given below in figure 3.



Note that the continuation is well defined for the stresses of the cycle during take-off, flight and landing for a particular lower wing station of the airframe. However, insofar as possible we shall leave the exact nature of $\mathcal{L}$ unspecified in order to obtain as much generality as possible.

Let Λ denote a random function taking values in $\mathcal{L}$ and Λ^i denote the restriction of Λ in domain to the interval $(0,i)$.

Again we call Λ^i the i^{th} (random) history of the load and it corresponds to λ^i as previously defined. The length of the cycle, call it M, may be a random integer. For $i \le M$, Λ^i is well defined and for $i > M$ we follow the convention of (3.7) and set $\Lambda^i = \omega_0$.

We define the random load $\underline{\Lambda}$ to be *cyclic in distribution* whenever

$$\underline{\Lambda} = \Lambda_1 * \Lambda_2 * \cdots$$

where Λ_j for $i = 1, 2, \ldots$ are independent and identically distributed replications of some random load function Λ on $\mathcal{E}$. Note in the case $M = m$ with probability one that we have equality in distribution for all $j \neq k$ of $\Lambda_j^i = \Lambda_k^i$ for each $i = 1, \ldots m$. This replaces the cyclic assumption of Section 3.

During the j^{th} cycle let $Z_j^i(\Lambda_j^i)$ be the random microscopic crack extension due both to that portion of the j^{th} random cycle up to the i^{th} fluctuation, namely Λ_j^i, and the stochastic variation within the material, i.e. its inhomogeneity.

We see the total damage in the sense of crack extension during the j^{th} cycle of load fluctuations, from our assumption, is

$$Y_j(\Lambda_j) = \sum_{i \geq 1} Z_j^i(\Lambda_j^i). \tag{4.4}$$

We now let

$$S_n(\underline{\Lambda}) = \sum_{j=1}^{n} Y_j(\Lambda_j) \tag{4.5}$$

be the total crack length at the end of n cycles. Note that $S_n(\underline{\Lambda})$, by assumption, is the sum of n independent and identically distributed random variables.

What we would like to prove is that (3.5) holds for the random variable $N_{Y(\underline{\Lambda})}$, which is defined by the formula corresponding to (3.2)

with S_n now defined by (4.5). However, we cannot proceed exactly as before because the assumption of $Z_j^i(\Lambda_j^i)$ being NBUE is not sufficient to imply that $Z_j^i(\Lambda_j^i)$ are in the same class. One may see this from the conditional expectation

$$P[Z_j^i(\Lambda_j^i) > x] = ER(x:\Lambda_j^i)$$

by knowing mixtures of NBUE variables are not necessarily NBUE. See the example in Appendix A. Moreover, the $Z^i(\Lambda^i)$ need not even be independent since the Λ^i are not. Therefore, the hope that the NBUE property could be invoked to use the right-hand inequality of (3.5) is vain. Consequently, we must impose further assumptions on the basic model.

We now state the crucial assumption

5° For any $x > 0$, a co-distribution $R(x:\cdot)$ is a convex function over the convex sample space $\mathcal{L}$.

Clearly $\mathcal{L}^+$ and $\mathcal{L}^-$ are convex spaces. They represent loading functions in which the specimen is exclusively in either compression or tension. However, spectra which contain fluctuations which are both compressive and extensional may not, in general, form a convex space unless other assumptions are made. Thus to assume that the spectra of the ground-air-ground cycle is a convex space it would be necessary to assume that each of the three portions would be of fixed length. Of course, this might be suitable for some applications but not for others.

Before we proceed with determining the implications of this assumption let us discuss its reasonableness. That $\mathcal{L}$ is a convex space is

merely a mathematical restriction on the type of admissible loading. It is made only to facilitate our analysis. However, the second part of the assumption, to wit , $R(x:\cdot)$ is a convex function, is a strong inference about the nature of fatigue crack growth. Thus any conclusion we draw must reflect any uncertainty that we have in this assumption.

Consider the vector representation of λ as given in (4.2) It is clear that for any $x > 0$, $R(x:\lambda)$ should be convex increasing as a function of each of the fluctuation peaks (maximum stress per oscillation) $\lambda(2i+1)$, $i=0,\ldots,\frac{m}{2}-1$ and convex decreasing as a function of the fluctuation troughs (minimum stress per oscillation) $\lambda(2j)$, $j=1,\ldots,\frac{m}{2}-1$. See References [3] and [9] and Appendix B.

What we assume in the second part of 4° is the joint convexity over all the variables $\lambda(i)$, $i=1,\ldots,m-1$ restricted by

$$\lambda(2j) < \lambda(2j+1) > \lambda(2j+2), \qquad j=0,\ldots,\frac{m}{2}-1.$$

Thus our simply stated assumption of joint convexity is not too great a conceptual step from the known convexity in each variable separately.

Because $\mathcal{L}$ is assumed to be a convex space it follows, since Λ is a random function taking values in $\mathcal{L}$, that $E\Lambda$ is a spectrum in $\mathcal{L}$. The other part of 5°, namely that $R(x:\cdot)$ is convex over $\mathcal{L}$ is equivalent with the stochastic inequality

$$Y(\Lambda) \geq Y(E\Lambda). \tag{4.6}$$

To see this note by the theorem on conditional probabilities that for all $y > 0$

(4.7) $$P[Y(\Lambda) \geq y] = ER(y:\Lambda) \geq R(y:E\Lambda) = P[Y(E\Lambda) \geq y].$$

By adding independent identically distributed variates each pair satisfying (4.6) we have from the definition in (4.7) the stochastic inequality

$$S_n(\underline{\Lambda}) \geq S_n(E\underline{\Lambda})$$

By analogy with the fundamental lemma equation (3.4) we have

$$[N_{Y(\Lambda)} \geq n] = [S_n(\underline{\Lambda}) \leq W]$$

thus there follows the stochastic inequality

(4.8) $$N_{Y(\Lambda)} \leq N_{Y(E\Lambda)}.$$

But note that $Y_j(E\Lambda)$ are NBUE by the assumption of Section 3. Thus by taking expectations of (4.8) and the Equation (3.5) of the fundamental lemma we have

(4.9) $$EN_{Y(\Lambda)} \leq \frac{EW}{\mu(E\Lambda)}.$$

For $i \geq 1$, $j=0,\ldots,k$ set

$$P[\Lambda^i = \omega_j] = p_j^i.$$

We consider the typical incremental crack growth per random cycle

$$Y(\Lambda) = \sum_{i \geq 1} Z^i(\Lambda^i)$$

then by properties of conditional expectation

(4.10) $$EZ^i(\Lambda^i) = E_{\Lambda^i} E[Z^i(\omega_j) | \Lambda^i = \omega_j] = \sum_{j=1}^{k} \mu_j p_j^i.$$

Thus we obtain the expression

$$\mu = EY(\Lambda) = \sum_{i\geq 1} \sum_{j=1}^{k} \mu_j p_j^i = \sum_{j=1}^{k} \mu_j \bar{n}_j$$

where

$$\bar{n}_j = \sum_{i\geq 1} p_j^i = E \sum_{i\geq 1} \{\Lambda^i \approx \omega_j\} = En_j(\Lambda) \tag{4.11}$$

is the expected number of histories Λ^i in the cycle Λ equivalent with ω_j, to use the notation of (3.10). By multiplying (3.9) by $\bar{n}_j$ and summing we have

$$w \sum_{1}^{k} \frac{\bar{n}_j}{\nu_j+1} \leq \mu \leq w \sum_{j=1}^{k} \frac{\bar{n}_j}{\nu_j} .$$

Rewriting the above we obtain

$$\frac{1}{\sum_{1}^{k} \frac{\bar{n}_j}{\nu_j}} \leq \frac{w}{\mu} \leq \frac{1}{\sum_{1}^{k} \frac{\bar{n}_j}{\nu_j+1}} .$$

By Lemma 1 we know that for every sequence of independent identically distributed random variables, in particular $Y_j(\Lambda_j)$, $j=1,2,\ldots,$ we have

$$\frac{w}{\mu} - 1 \leq EN_{Y(\Lambda)} .$$

Thus we obtain

$$\frac{1}{\sum_{j=1}^{k} \frac{\bar{n}_j}{\nu_j}} - 1 \leq EN_{Y(\Lambda)} .$$

It is known that for some MNBUE variables Y, we have $EN_Y \nleq w/EY$. We must make use of the convexity as given by (4.6).

Combining the left-hand side of (3.12) with (4.9) we have

$$EN_{Y(\Lambda)} \le \frac{w}{\mu(E\Lambda)} \le \frac{1}{\sum_{1}^{k} \frac{n_j(E\Lambda)}{\nu_j+1}} .$$

This completes the result for random spectra. Note that if Λ is deterministic this reduces to the same bounds which were given in (3.13).

<u>Theorem 2</u>: <u>For Random Spectra on Convex Spaces</u>. If the stochastic nature of incremental crack growth satisfies assumptions 1°, 2° and 5°, and each independent replication of a random spectra Λ with convex sample space results in independent crack extensions $Y(\Lambda)$, then the number of replications of Λ that can be made until the crack exceeds a given stochastic limit, satisfying 4°, has finite expectation $EN_{Y(\Lambda)}$ which is bounded by

$$\text{(4.12)} \qquad \frac{1}{\sum_{j=1}^{k} \frac{En_j(\Lambda)}{\nu_j}} - 1 \le EN_{Y(\Lambda)} \le \frac{1}{\sum_{j=1}^{k} \frac{n_j(E\Lambda)}{\nu_j+1}} ,$$

where by Assumption 3° for each $\omega_j \in \Omega$ we know

$$\text{(4.13)} \qquad \nu_j = EN_{Y(\omega_j)} \quad \text{and} \quad n_j(\lambda) = \sum_{i\geq 1} \{\lambda^i \simeq \omega_j\} .$$

We can also obtain the

Corollary 2.1: Supressing assumption 5°, including the convexity of the sample space of the random spectra Λ, we see that the number of replications of Λ until the crack exceeds the stochastic limit satisfying 4° has expectation bounded below by the left hand side of equation (4.12).

5. Conditionally Convex Spaces of Random Spectra

Let us now consider the situation when the loading spectra may have its length, or a portion, of random duration. This case cannot be treated by the assumptions used hithertofore. For example, each portion of the ground-air-ground cycle such as take-off, flight and landing may well be of such significant variability that it should be considered of random duration. In this situation the sample space of load spectra would not be convex. Nevertheless, there is a set of conditions which if known (for example, the gross take-off weight, barometric pressure, and the length of time of flight) make all the random spectra with these given boundary conditions have the same distribution on a convex subset of the sample space.

Thus we postulate, there exists a random couple (Λ,Φ) having a joint distribution on $\mathcal{L} \times \mathcal{T}$ for which the conditional random function $\Lambda|\Phi$, for each $\phi \in \mathcal{T}$, is a random spectrum taking values in a convex subspace of $\mathcal{L}$. The case we have in mind is $\Lambda|\phi$ being a random spectrum of fixed length in each of its portions as was considered in the preceding section.

What we wish to determine is bounds on

$$EN_{Y(\Lambda,\Phi)} \quad \text{where} \quad Y(\Lambda,\Phi) = \sum_{i\geq 1} Z^i(\Lambda^i,\Phi).$$

Here again $Z^i(\Lambda^i,\Phi)$ are the random incremental crack extensions immediately following the i^{th} fluctuation within the history.

By assumption $Y_j(\Lambda_j,\Phi_j)$, $j=1,2,\ldots$, are independent and identically distributed random variables with expectation $\mu = EY(\Lambda,\Phi)$.

Rearranging terms using the properties of conditional expectation, and with an obvious modification of (4.10), we have

$$\mu = E_{\Phi} \sum_{i\geq 1} EZ^{i}(\cdot^{i}|\Phi) = \sum_{j=1}^{k} \mu_j E_{\Phi} \sum_{i\geq 1} p_j^{i}(\Phi) .$$

We now have

$$\mu = \sum_{j=1}^{k} \mu_j E_{\Phi} E_{\Lambda} n \ (\Lambda|\Phi) = \sum_{j=1}^{k} \mu_j n_j^*$$

where we set, by analogy with (3.10) and (4.11)

$$n_j^* = En_j(\Lambda,\Phi) \qquad j = 1,\ldots,k$$

Proceeding as before, from (3.9), we obtain

$$\frac{1}{\sum_{j=1}^{k} \frac{n_j^*}{\nu_j}} - \frac{w}{\mu} - \frac{1}{\sum_{j=1}^{k} \frac{n_j^*}{\nu_j+1}}$$

By the fundamental lemma, for any non-negative random variable

$$\frac{1}{\sum_{j=1}^{k} \frac{n_j^*}{\nu_j}} - 1 \leq \frac{w}{\mu} - 1 \leq EN_{Y(\Lambda,\Phi)}.$$

Consider the set

$$\mathcal{P}^* = \{\phi \in \mathcal{P}: ER(\cdot:E\Lambda|\Phi) \geq R(\cdot:E\Lambda|\phi)\}$$

where an inequality between functions indicates the corresponding inequality between functional values for all values of the domain. If $\mathcal{P}^*$ is not empty, it follows by Zorn's lemma, since linearly ordered subsets of $\mathcal{P}^*$, ordered by the function $R(\cdot:E\Lambda|\phi)$ for $\phi \in \mathcal{P}^*$, have lower bounds, that there are minimal elements of $\mathcal{P}^*$. We then pick $\hat{\phi}$ as the maximum likelihood of the minimal elements rated by the marginal density of Φ.

Proceeding in the same manner as before, by using the properties of conditional expectation and the convexity of the domain as well as that of the co-distribution function, we have

$$ER(\cdot:\Lambda,\Phi) \geq ER(\cdot:E\Lambda|\Phi) \geq R(\cdot:E\Lambda|\hat{\Phi}).$$

Ultimately we obtain

$$(5.1) \qquad \frac{1}{\sum_{j=1}^{k} \frac{En_j(\Lambda,\Phi)}{\nu_j}} - 1 \leq EN_{Y(\Lambda,\Phi)} \leq \frac{1}{\sum_{j=1}^{k} \frac{n_j(E\Lambda|\hat{\Phi})}{\nu_j+1}}$$

<u>Theorem 3: For Random Spectra on Conditionally Convex Spaces</u>. If the stochastic nature of incremental crack growth satisfies assumptions 1°, 2° and 3° and each independent replication of the random couple (Λ,Φ) causes an independent crack extension $Y(\Lambda,\Phi)$ and the conditional spectrum $\Lambda|\phi$ has a convex sample space for each ϕ and the marginal distribution is known, then the number of replications of (Λ,Φ) that can be made until the crack exceeds a given stochastic limit, satisfying 4°, has finite expectation $EN_{Y(\Lambda,\Phi)}$ which is bounded by (5.1) above.

APPENDIX A

Take $R(x:\lambda) = \exp\{-x/\mu(\lambda)\}$ for $x > 0$. One checks easily that the NBUE property is satisfied. However, let Λ take two values, say λ_1 and λ_2, each with equal probability. Then

$$ER(x:\Lambda) = \tfrac{1}{2}[\exp(-x/\mu_1) + \exp(-x/\mu_2)] \; .$$

To have this mixed distribution be NBUE is equivalent with

$$\tfrac{1}{2}(\mu_1+\mu_2) \geq \frac{\int_0^\infty \exp[\frac{x+y}{-\mu_1}] + \exp[\frac{x+y}{-\mu_2}] \, dy}{\exp(-x/\mu_1) + \exp(-x/\mu_2)}$$

for all $x > 0$. Upon simplifying we obtain the inequality

$$(\mu_1-\mu_2)\, e^{-x/\mu_2} \geq (\mu_1-\mu_2)\, e^{-x/\mu_1}$$

which is clearly impossible unless $\mu_1 = \mu_2$.

APPENDIX B

In order to see that current hypotheses concerning crack growth rates are not in disagreement with this assumption we note that $R(x:\cdot)$ being convex over $\mathcal{L}$ for each $x > 0$ implies that $\int_0^\infty R(x:\cdot)dx$ is also convex over $\mathcal{L}$. This integral is the expected crack extension. In [5] the "crack growth rate" is given as proportional to

$$f(x,y) = x^a(x-y)^b \quad \text{for} \quad x > y > 0$$

where x is the maximum load and y is the minimum load and both are unscaled. We interpret these two concepts as being the same.

The question now is, does there exist a region of values of a,b over which f is convex and if so, is the region in conformity with those values obtained by experiment.

A well known condition both necessary and sufficient for the convexity of twice differentiable functions is

$$f_{xx} \geq 0, \quad f_{yy} \geq 0, \quad f_{xx}f_{yy} \geq f_{xy}^2,$$

with subscripts denoting partial differentiation. One checks that

$$f_{xx}/f = \frac{a^2-a}{x^2} + \frac{b^2-b}{(x-y)^2} + \frac{2ab}{x(x-y)}$$

$$f_{yy}/f = \frac{b^2-b}{(x-y)^2}$$

$$f_{xy}/f = \frac{-ab}{x(x-y)} \quad .$$

Thus clearly $a,b \geq 1$ implies $f_{xx} \geq 0$, $f_{yy} \geq 0$.

We now check that $f_{xx}f_{yy} \geq f_{xy}^2$ is equivalent with $f_{xx}/f \geq a^2b^2/[x^2(b^2-b)]$. But note that $xy\sqrt{f_{xx}/f} \geq y\sqrt{a^2-a} + x\sqrt{b^2-b}$ since $1 \leq b + a$ implies $(a^2-a)(b^2-b) \leq a^2b^2$. Thus

$$y\sqrt{a^2-a} + x\sqrt{b^2-b} \geq yab/\sqrt{b^2-b}$$

is a sufficient condition for convexity. However this last inequality is equivalent with

$$\frac{y}{x} \geq 1 - \frac{b^2-b}{ab-\sqrt{(b^2-b)a^2-a}}$$

Now $1 \geq \frac{y}{x} \geq 0$, hence the inequality above is true if the right hand side is always negative, which is true if

$$1 \leq \frac{b}{a} - \frac{1}{a} + \sqrt{(1-\frac{1}{b})(1-\frac{1}{a})} \ .$$

But the square root of a number less than one exceeds the number. Hence, a sufficient condition for convexity is

$$1 \leq \frac{b}{a} - \frac{1}{a} + (1-\frac{1}{b})(1-\frac{1}{a}) \quad \text{if an only if} \quad b \geq 1 + \sqrt{a}\ .$$

Thus if $b = 4$, $a \leq 9$ and if $b = 3$, $a \leq 4$. These values are not in disagreement with observed values. Also note that in order to obtain simplicity we have been exceedingly wasteful with our inequality. For example, from physical reasons x must be bounded and this restriction has not been imposed.

CONCLUDING REMARKS

In this paper we have made an attempt to find physically realizable conditions under which both upper and lower bounds on the expected number of cycles until failure can be found and expressed by a formula which is a generalization of a variant of Miner's rule.

We should not lose sight of the fact that some of the mathematical structure imposed was only necessary to obtain the upper bound and thus insure that the expected number of cycles is given approximately by one form of Miner's rule. Without making the restrictive assumption 5°, we have the lower bound (4.11) holding for arbitrary $\mathcal{L}$.

Thus we conclude that a variation of Miner's rule gives a conservative lower bound for the expected number of cycles until failure under very general conditions.

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